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Convergence of Galerkin solutions and continuous dependence on data in spectrally-hyperviscous models of 3D turbulent flow

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ABSTRACT

We obtain results on the convergence of Galerkin solutions and continuous dependence on data for the spectrally-hyperviscous Navier–Stokes equations. Let u_N denote the Galerkin approximates to the solution u, and let $w_N = u - u_N$. Then our main result uses the decomposition $w_N = P_n w_N + Q_n w_N$ where (for fixed n) P_n is the projection onto the first n eigenspaces of $A = -\Delta$ and $Q_n = I - P_n$. For assumptions on n that compare well with those in related previous results, the convergence of $\|Q_n w_N(t)\|_{H^\beta}$ as $N \to \infty$ depends linearly on key parameters (and on negative powers of λ_n), thus reflective of Kolmogorov-theory predictions that in high wavenumber modes viscous (i.e. linear) effects dominate. Meanwhile $\|P_n w_N(t)\|_{H^\beta}$ satisfies a more standard exponential estimate, with positive, but fractional, dependence on λ_n. Modifications of these estimates demonstrate continuous dependence on data.

1. Introduction

We obtain strong convergence results for Galerkin-projection approximations and estimates that establish continuous dependence on data for solutions of the 3D spectrally-hyperviscous Navier–Stokes equations (SHNSE):

$$u_t + \nu A u + \mu A_\varphi u + (u \cdot \nabla) u + \nabla p = g, \tag{1.1a}$$

$$\nabla \cdot u = 0. \tag{1.1b}$$

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Here $A = -\Delta$, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, $g = (g_1, g_2, g_3)$ is the external force, and p is the pressure. We have that $u_i = u(x,t)$, $g_i = g_i(x,t)$, $i = 1, 2, 3$, and $p = p(x,t)$ where $x \in \Omega$, a domain in $\mathbb{R}^3$.

We assume that Ω is a periodic box, then "moding out" the constant vectors as in standard practice, A has eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ with corresponding eigenspaces $E_1, E_2, \ldots$; let P_m be the projection onto $E_1 \oplus \cdots \oplus E_m$, let $Q_m = I - P_m$ and let P_{E_j} be the projection onto each E_j. The general class of operators A_φ that we consider satisfy $A_\varphi = \sum_{j=1}^{\infty} a(\lambda_j) P_{E_j}$ such that (for a constant $\alpha > 1$) $A_\varphi \geqslant A_m \equiv Q_m A^\alpha$ in the sense of quadratic forms, i.e., $(A_\varphi v, v) \geqslant (A_m v, v)$.

Particular distributions of the coefficient $a(\lambda)$ that we have in mind come from certain subgrid-scale (SGS) closure models for the Navier–Stokes equation (NSE), the NSE being (1.1) with $\mu \equiv 0$. In particular spectral eddy-viscosity (SEV) techniques suggest an Arrhenius-like exponential dependence of $a(\lambda)$ on λ, so that $a(\lambda)$ is small for low to intermediate wavenumbers and then rises rapidly as λ approaches the limit of the resolvable wavenumbers (see [4,5,25] for further discussion and references). Though originally conceived with second-order operators in mind, the use of hyperviscosity to model the rapid rise of $a(\lambda)$ for large λ was suggested in [8]. In the spectral vanishing viscosity (SVV) method, the distribution of $a(\lambda)$ is similar to the SEV case, but $a(\lambda)$ is identically zero for low wavenumbers; this corresponds to a distinguished class of A_φ in which $a(\lambda_j) = 0$ for $j \leqslant m_0 \leqslant m$, $0 \leqslant a(\lambda_j) \leqslant \lambda_j^\alpha$ for $m_0 \leqslant j \leqslant m$, and $a_j(\lambda_j) \geqslant \lambda_j^\alpha$ for $j \geqslant m+1$, where m_0 stays reasonably close to m. Indeed, "SVV can be thought of as using hyperviscous dissipation that will affect only the high Fourier modes" [23]. Typically SVV is implemented with second-order operators or kernels for computational simplicity, but the full realization of hyperviscous SVV has been contemplated, e.g. by using discontinuous Galerkin methods as suggested in [23]. Both the SEV and SVV versions of $a(\lambda)$ are included in the assumption $A_\varphi \geqslant Q_m A^\alpha$, where m is bounded above by the limit of the resolvable wavenumbers. The direct practical effect, particularly in the SVV case, is to enforce the rapid decay of high wavenumbers, as predicted by the Kolmogorov theory [24], while preserving spectral accuracy and in particular the detailed features of the inertial range.

In [19] the NSE with spectral hyperviscosity was first studied theoretically and therein regularity results were obtained and it was argued that the model (1.1) improves spectral accuracy and regularizing properties compared with SEV. In [1] we adapted elements of the "CFT" framework [9,11,12, 32,33], and in particular applied the generalized Lieb–Thirring inequalities developed in [32,33], to obtain estimates on the dimension of the attractor for (1.1). These results compare well with those obtained for the NS-α model ([13]; see also [20,21]). Additionally in [1] we adapted the machinery developed in [16,17] to establish the existence of an inertial manifold of dimension m_0 for typical A_φ in the distinguished-class case with $\alpha \geqslant 3/2$, which implies that for m_0 large enough eigenmodes free of hyperviscosity control the essential dynamics.

Regarding in particular the attractor results in [1], let $\dim_F \mathcal{A}$ and $\dim_H \mathcal{A}$ be respectively the fractal and Hausdorff dimensions of the attractor $\mathcal{A}$, then we demonstrated that $\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant K m^a \kappa_d^b$ where κ_d is the (normalized) Kolmogorov wavenumber, $K = c K_\alpha(\alpha)(\nu/\mu)^{9/(10\alpha)}$ where $K_\alpha(\alpha)$ is on the order of magnitude of unity, c is a dimensionless constant depending only on the shape (but not the size) of Ω, $(\nu/\mu)^{9/(10\alpha)}$ is of manageable size as discussed in [1], $a = 3(\alpha-1)/(5\alpha)$ is a fractional power, and $b = (6\alpha+9)/(5\alpha) < 3$. In particular we found that $\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant K m^a \kappa_d^b \leqslant K \kappa_d^3$ for $m \leqslant \kappa_d^3$, i.e. for m so large as to suggest machine-indistinguishability from NSE solutions. This robust conformance with the Landau–Lifschitz estimates [28] appears to be unique among NSE closure models, and to achieve this kind of behavior the NSE only needs to be regularized in the highest modes; meanwhile b is significantly lower for more realistic choices of m, implying the ability of the model (1.1) to reduce the number of degrees of freedom in spectrally-accurate turbulence simulation.

With simulation in mind as well as further exploration of the finite-dimensional character of (1.1) we study here convergence results of Galerkin approximations to (1.1); as corollaries of our estimates we will also obtain results on the continuous dependence on data. Let P_N be the projection onto $E_1 \oplus \cdots \oplus E_N$ for some $N > m$, then the Galerkin approximation u_N to u satisfies

$$(u_N)_t + \nu A u_N + \mu A_\varphi u_N + P_N(u_N \cdot \nabla)u_N + \nabla p_N = g_N, \tag{1.2a}$$

$$\nabla \cdot u_N = 0 \tag{1.2b}$$

where $g_N = P_N g$ and for initial data we take $u_N(x,0) = P_N u(x,0)$. Let $w_N = u - u_N$, let $G_N = g - g_N$, and let L_g and U_g be defined as in (2.8) and (2.12) below, then our goal in this paper is to establish estimates that show that $w_N \to 0$ strongly as $N \to \infty$. Our first result in this direction uses a more standard approach but nonetheless generalizes results in [35] applicable to (1.1) with $m = 0$.

Theorem 1. *Let $T > 0$, then for any $\beta \geqslant 1$ and for some $\theta \geqslant 1/2$ we have for all $0 \leqslant t \leqslant T$ that if $\alpha \geqslant 3/2$*

$$\left\| A^{\beta/2} w_N(t) \right\|_2^2 \leqslant W_N(t) e^{\frac{4}{\mu}\lambda_m^{\alpha-1} C_0 C_{0,1} T} \tag{1.3a}$$

where C_0 is a generic constant and $C_{0,1}$ is a polynomial in U_g and T of degree depending on θ. If $\alpha - 3 \leqslant \beta \leqslant \alpha$ then for $\gamma = (\alpha - \beta)/2$ we obtain

$$\left\| A^{\beta/2} w_N(t) \right\|_2^2 \leqslant W_N(t) \exp\left(\frac{8}{\mu} \lambda_m^{\alpha-1} \frac{C_0}{(\nu \lambda_1^2)} \int_0^t \|g\|_2^2 \, ds \right). \tag{1.3b}$$

Here

$$W_N(t) \equiv \left\| A^{\beta/2} w_N(0) \right\|_2^2 + \frac{4\lambda_m^{\alpha-1}}{\mu} \int_0^t \left(\left\| A^{-\gamma} G_N \right\|_2^2 + \left\| A^{-\gamma} Q_N(u \cdot \nabla)u \right\|_2^2 \right) ds. \tag{1.3c}$$

That (1.3c) implies uniform convergence of $\|A^{\beta/2} w_N(t)\|_2$ to zero on each $[0, T]$ can be seen by using the Dominated Convergence Theorem. In particular for the term $\|A^{-\gamma} Q_N(u \cdot \nabla)u\|_2^2$ in the case $1 \leqslant \beta \leqslant \alpha$ we use (2.13a) below to show that there is a generic constant M_0 such that

$$\begin{aligned} \left\| A^{-\gamma} Q_N(u \cdot \nabla)u \right\|_2^2 &= \left\| Q_N A^{-\gamma}(u \cdot \nabla)u \right\|_2^2 \\ &\leqslant \left\| A^{-\gamma}(u \cdot \nabla)u \right\|_2^2 \leqslant M_0 \|\nabla u\|_2^2 \left\| A^{\beta/2} u \right\|_2^2. \end{aligned} \tag{1.4}$$

Now $\|A^{\beta/2} u\|_2^2$ is uniformly bounded for any β by the remarks on regularity in [1, Section 2], and

$$\nu \int_0^t \|\nabla u\|_2^2 \, ds \leqslant \|u_0\|_2^2 + \frac{1}{\nu \lambda_1} \int_0^t \|g\|_2^2 \, ds \tag{1.5}$$

by the appropriate modification of the standard energy inequality (see (2.6) below). For $\beta \geqslant \alpha$ we replace $\|\nabla u\|_2^2$ by $\|A^\theta u\|_2$ on the right-hand side of (1.4) via (2.15b) below and use the uniform boundedness of $\|A^\theta u\|_2$ from [1] as noted. Meanwhile $\|G_N\|_2 \leqslant \|g_N\|_2 + \|g\|_2 \leqslant 2\|g\|_2$, thus by the Dominated Convergence Theorem, since both the left-hand side of (1.4) and $\|G_N\|_2^2$ go to zero as $N \to \infty$ for each t, we have that (1.3c) goes to zero as $N \to \infty$.

We remark that the results in [2] together with the regularity discussion in [1] noted above show that solutions to (1.1) are globally regular for any divergence-free $u_0 \in H^{\beta/2}$ and any positive β. These results generalize the classical results in [29] for the case $m = 0$ in which global regularity is demonstrated for $\beta/2 = \alpha \geqslant 5/4$. We have assumed $\beta \geqslant 1$ for greater simplicity in proving some of our foundational estimates and for utility of application in stating our results; in particular the case $\beta = 1$

is a common assumption in analytical studies of the NSE. Since the remarks on regularity in [1, Section 2] show that the solutions for all positive times are in $H^{\beta_1/2}$ for any $\beta_1 > 0$, our theorems can be adapted to hold for less regular initial data by replacing u_0 by $u(x,\delta)$ for any $\delta > 0$.

In the case of decaying turbulence (DT), which we define as the condition

$$g \in L^2\big([0,\infty]; L^2(\Omega)\big), \tag{1.6}$$

let

$$G_\infty = \int_0^\infty \|g\|_2^2 \, ds, \tag{1.7}$$

then the following follows immediately from Theorem 1 using the dominated convergence arguments above for $\beta \leqslant \alpha$.

Theorem 2. *Suppose g is in the DT case* (1.6)*, and suppose that* $\alpha \geqslant 3/2$ *and* $\beta \leqslant \alpha$. *Then for* $W_N(t)$ *as in* (1.3c) *we have for each* $t \geqslant 0$ *that*

$$\big\| A^{\beta/2} w_N(t) \big\|_2^2 \leqslant W_N(t) e^{\frac{8}{\mu}\lambda_m^{\alpha-1} C_0(\nu\lambda_1)^2 G_\infty} \tag{1.8}$$

and so by the remarks above the convergence $u_N \to u$ *in* H^β *is uniform on* $[0,\infty)$ *as* $N \to \infty$.

In [35], C^1-convergence of Galerkin solutions was established on each $[0,T]$ for a general class of semilinear parabolic PDE which include the HNSE, i.e. (1.1) with $m = 0$. In addition to considering arbitrary m here, we can allow for arbitrarily-sized initial data while in [35] $P_N u_0$ needs to be in a compact trapping region. Since in Theorem 1 $\beta > 0$ is arbitrary, we have convergence in C^k for all k by the Sobolev embedding theorems. In the case of decaying turbulence as noted, we obtain uniform H^β-convergence on all of $[0,\infty)$. Finally, the estimates given by (1.3a)–(1.3c) depend explicitly on generic constants and the data.

Before discussing our next convergence results we note the following additional property of the DT case:

Theorem 3. *Let* (1.1) *be such that g is in the DT case, then*

$$\big\| A^{\beta/2} u(t) \big\|_2 \to 0 \quad \text{as } t \to \infty \tag{1.9}$$

for all $\beta > 0$.

This is also a simple consequence of the assumption (1.6), though slightly less immediate than Theorem 2; we prove it in Section 2 below.

The results in Theorem 1 can be improved in the case $\beta \leqslant \alpha$ by using a spectral decomposition approach. This was the strategy in [1, Section 2], wherein regularity bounds for $Q_m u$ were found depending on negative powers of m. The techniques used can be adapted to apply similarly to $Q_n u$ for any $n \geqslant m$, thus showing that the effects of the high wavenumber modes are minimized for large enough n (here $Q_n = I - P_n$ where P_n projects onto $E_1 \oplus \cdots \oplus E_n$). Such results roughly reflect the Kolmogorov theory of turbulence [24], which predicts that the high wavenumber modes decay rapidly and quickly become of no dynamical consequence. Additionally the theory predicts that the dynamics in the high wavenumber modes are dominated by the (linear) effects of viscosity. In what follows we will seek as much as possible to achieve convergence estimates for the high wavenumber modes that become minimal for large enough n while depending on linear combinations of the data and the input from the low wavenumber modes.

We will impose conditions on n independently of m other than the restriction $n \geqslant m$ and thus allow the model to be fixed while studying the convergence behavior of $Q_n w_N$ and $P_n w_N$ for various choices of n. Since $N \to \infty$ we can without loss of generality assume that $N > n$. In analyzing the convergence of $P_n w_N$ and $Q_n w_N$ separately in the next theorems, our techniques will in part resemble those developed for the determining-modes theory ([10,15,22], also see [14, Chapter III]). But to reduce the exponential dependence on data new techniques will need to be constructed.

Let $\eta \geqslant 0$ be such that

$$\|u_0\|_2^2 \leqslant (1+\eta)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{1.10}$$

and let

$$\rho_P(t) = \sup_{0\leqslant s\leqslant t} \left\| A^{\beta/2} P_n w_N(s)\right\|_2^2, \tag{1.11}$$

$$\rho_Q(t) = \sup_{0\leqslant s\leqslant t} \left\| A^{\beta/2} Q_n w_N(s)\right\|_2^2, \tag{1.12}$$

$$F_{Q,N}(t) = \frac{4}{\mu}\|Q_n G_N\|_2^2 + \frac{4}{\lambda_1^{\gamma}}\left\|A^{-\gamma} Q_N(u\cdot\nabla)u\right\|_2^2, \tag{1.13}$$

and

$$\mathcal{F}_{Q,N}(t) = \sup_{0\leqslant s\leqslant t} \int_0^t e^{-\mu(s-\tau)\lambda_{n+1}^{\alpha}} F_{Q,N}(\tau)\,d\tau, \tag{1.14}$$

then for n chosen large enough the next result achieves estimates on $\|A^{\beta/2}Q_n w_N(t)\|_2^2$ that depend on linear combinations of terms involving $\rho_P(t)$, $A^{\beta/2}Q_n w(0)$, $Q_n G_N$, and $Q_N(u\cdot\nabla)u$.

Theorem 4. *Assume that $\alpha \geqslant 3/2 \geqslant \beta \geqslant 1$. Let L_g, η, $\rho_P(t)$, $\rho_Q(t)$, and $\mathcal{F}_{Q,N}$ be as above, then if we assume that*

$$\lambda_{n+1}^{2\alpha-5/2} \geqslant \frac{16C_1}{\mu^2}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{1.15}$$

we have for a generic constant C_1 that

$$\rho_Q(t) \leqslant 2\left\|A^{\beta/2}Q_n w_N(0)\right\|_2^2 + 2\mathcal{F}_{Q,N}(t) + \frac{16C_1}{\mu^2\lambda_{n+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_P(t). \tag{1.16}$$

The estimate (1.16) improves for larger n, and the convergence is uniform on any interval for which uniform convergence holds for $\rho_P(t)$. Just as significantly, we see that in some sense the convergence of $Q_n w_N$ to zero is controlled in large part by the convergence of $P_n w_N$, in analogy with results on determining modes (DM) for the NSE and with the inertial-manifold (IM) results in [1,16,17].

It is thus useful to compare our conditions on n with similar conditions imposed in the DM and IM results. The condition (1.15) basically requires that $\lambda_{n+1}^{\alpha-5/4} > c_1(\nu/\mu)G$ where $G = L_g/(\nu^2\lambda_1^{3/4})$ is the Grashoff number and $c_1 = 4C_1^{1/2}\lambda_1^{1/4}[5+3\eta+(\nu/\mu)(\lambda_1/\lambda_{n+1}^{\alpha})]^{1/2}$ is a generic constant (as will be

$c_2, c_3, \ldots$ in what follows). Our best estimates on the dimension of the inertial manifolds constructed in [1] were of the form $\lambda_n > c_2 G^2$ when $\alpha \geqslant 5/2$; such a lower bound is improved here for any $\alpha > 7/4$ and further improves as α grows. Using $\lambda_n \sim c\lambda_1 n^{2/3}$ the lower-bound condition is satisfied if $n^{(2/3)\alpha-5/6} > c_3(\nu/\mu)G$. This gives $n > c_3(\nu/\mu)^2 G^2$ when $\alpha \geqslant 2$, comparable to the 2D no-slip estimate for determining modes $n > c_4 G^2$ in [15]. For $\alpha \geqslant 11/4$ we match the condition $n > c_5 G$ derived in [22] for the 2D periodic case, in particular we have $n > c_6(\nu/\mu)^{6/7} G^{6/7}$ for $\alpha = 3$, and $n > c_7(\nu/\mu)^{6/11} G^{6/11}$ for $\alpha = 4$, used in [8] for $m = 0$. The Grashoff number is an upper bound for $[l_0/l_\epsilon]^2$; assuming that this is a sharp upper bound makes $G^{1/2}$ the wavenumber boundary of the inertial range. We cross this boundary as α passes $17/4$; values as high as $\alpha = 8$ were used in [3,4] in applications of (1.1) when $m = 0$.

Note that under the minimum conditions on n in Theorem 4 we have the condition that

$$\rho_Q(t) \leqslant 2\left\| A^{\beta/2} Q_n w_N(0)\right\|_2^2 + 2\mathcal{F}_{Q,N}(t) + \rho_P(t). \tag{1.17}$$

This simple expression can be used to obtain an estimate on $\rho_P(t)$ that improves the estimate in Theorem 1 and depends only on the data on the right-hand side of (1.17) and on U_g.

Theorem 5. *Let C_0, L_g, η, and $\mathcal{F}_{Q,N}$ be as above, assume that $\alpha \geqslant 3/2 \geqslant \beta \geqslant 1$, and let*

$$\mathcal{L}_{G_N} \equiv \sup_{0\leqslant s\leqslant t} \frac{3}{\nu\lambda_1} \int_0^s e^{-\nu\lambda_1(s-\tau)} \left\| G_N(\tau)\right\|_2^2 d\tau \tag{1.18}$$

then we have on each interval $[0, T]$ that for $w_{N,0} = w_N(0)$, $G_0 = L_g/(\nu\lambda_1)$, and for

$$W_{N,0}(t) \equiv \left\| A^{\beta/2} P_n w_{N,0}\right\|_2^2 + \frac{3\mathcal{L}_{G_N}}{(\nu\lambda_1)^2} + \frac{6\lambda_n^{1/2} C_0(3+2\eta+\nu)}{\nu^2} G_0^2\left(\left\| A^{\beta/2} Q_n w_{N,0}\right\|_2^2 + \mathcal{F}_{Q,N}(t)\right) \tag{1.19}$$

we have that

$$\rho_P(t) \leqslant W_{N,0}(t) \exp\left(\frac{12\lambda_n^{1/2} C_0}{\nu^2}\left[(1+\eta)G_0^2 + \frac{1}{\nu\lambda_1}\int_0^t \|g\|_2^2\, ds\right]\right) \tag{1.20}$$

and in the case of decaying turbulence we can replace $\int_0^t \|g\|_2^2\, ds$ by $G_\infty \equiv \int_0^\infty \|g\|_2^2\, ds$ and have uniform convergence for all $t \geqslant 0$.

Note the fractional dependence on n in both (1.19) and (1.20); in fact, since $\lambda_n \sim c\lambda_1 n^{2/3}$, this dependence is like $n^{1/3}$. We will discuss further the significance of Theorem 4 in the conclusion. We will also discuss in greater detail the background behind the SEV and SVV models to give additional context to the results presented here. Meanwhile, after making some preliminary observations and calculations in the next section, as well as proving Theorem 3, we will prove Theorem 1 (and thus Theorem 2) in Section 3. In Section 4 we will prove Theorems 4 and 5, and in Section 5 we will discuss results which adapt the techniques of Theorems 4 and 5 to obtain continuous dependence on data in the context of spectral decomposition.

2. Preliminaries and proof of Theorem 2

We express the Sobolev inequalities on Ω in terms of the operator $A = -\Delta$:

$$\|\upsilon\|_q \leqslant M_1 \left\| A^\theta \upsilon \right\|_p \tag{2.1}$$

where $q \leqslant 3p/(3-2\theta p)$ and $M_1 = M_1(\theta, p, q, \Omega)$. For the semigroup $\exp(-tA)$ we have the decay estimate

$$\left\| e^{-tA}\upsilon \right\|_2 \leqslant \|\upsilon\|_2 e^{-\lambda_1 t} \tag{2.2}$$

and, since A is analytic there is a constant c_2 such that

$$\left\| A^\beta e^{-tA}\upsilon \right\|_2 \leqslant c_2 t^{-\beta} \|\upsilon\|_2 \tag{2.3}$$

for any $\beta > 0$ where A^β is defined by $A^\beta = \sum_{j=1}^{\infty} \lambda_n^\beta P_{E_j}$ where as above P_{E_j} is the projection onto the jth eigenspace. Like the standard NSE, (1.1) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.1) with u we have that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu \left\| A^{1/2}u \right\|_2^2 + \mu \left\| A_\varphi^{1/2} u \right\|_2^2 = (g, u) \tag{2.4}$$

noting that since $\operatorname{div} u = 0$ we have that $(\nabla p, u) = 0$ and $((u\cdot\nabla)u, u) = -((\nabla\cdot u)u, u) = 0$.

Now

$$(g, u) = \left(A^{-1/2}g, A^{1/2}u\right) \leqslant \frac{\nu}{2}\left\| A^{1/2}u \right\|_2^2 + \frac{1}{2\nu}\left\| A^{-1/2}g \right\|_2^2; \tag{2.5}$$

combining (2.5) with (2.4) using $A_\varphi \geqslant Q_m A^\alpha$ and multiplying by 2 we have our basic energy inequality

$$\frac{d}{dt}\|u\|_2^2 + \nu \left\| A^{1/2}u \right\|_2^2 + 2\mu \left\| Q_m A^\alpha u \right\|_2^2 \leqslant \frac{1}{\nu\lambda_1}\|g\|_2^2 \tag{2.6}$$

where we note that by Poincaré's inequality $\|A^{-1/2}g\|_2 \leqslant \lambda_1^{-1/2}\|g\|_2$; note that (2.6) reduces to the standard NSE energy inequality when $\mu = 0$. We will use 2 consequences of (2.6), the first obtained by discarding the term $u\|A_\varphi^{1/2}u\|_2^2$ and again using Poincaré to obtain

$$\frac{d}{dt}\|u\|_2^2 + \nu\lambda_1\|u\|_2^2 \leqslant \frac{1}{\nu\lambda_1}\|g\|_2^2 \tag{2.7}$$

so that, setting

$$L_g = \sup_{t\geqslant 0}\|g\|_2^2 \tag{2.8}$$

we have that

$$\frac{d}{dt}\|u\|_2^2 + \nu\lambda_1\|u\|_2^2 \leqslant \frac{L_g^2}{\nu\lambda_1}. \tag{2.9}$$

Solving the differential inequality (2.9) we have that for $u_0 = u(x,0)$

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \int_0^t \left(\frac{L_g^2}{\nu\lambda_1}\right) e^{-\nu\lambda_1(t-s)}\,ds \tag{2.10}$$

or, since $L_g^2/(\nu\lambda_1)$ is a constant,

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \left(\frac{L_g}{\nu\lambda_1}\right)^2. \tag{2.11}$$

Thus we have the a priori estimate

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 + \left(\frac{L_g}{\nu\lambda_1}\right)^2 \equiv U_g^2. \tag{2.12}$$

Next we prove the following technical lemma which will be used several times in the sections to follow:

Lemma 6. *Let $v \in H^1(\Omega)$, let $w \in H^{\beta/2}(\Omega)$, and suppose that $\alpha \geqslant 3/2$, $\alpha \geqslant \beta$, and $\beta \geqslant 1$. Then for $\gamma = (\alpha-\beta)/2$ there exist constants M_0 and M_0' such that*

$$\|A^{-\gamma}(v\cdot\nabla)w\|_2 \leqslant M_0\|\nabla v\|_2\|A^{\beta/2}w\|_2 \tag{2.13a}$$

and

$$\|A^{-\gamma}(w\cdot\nabla)v\|_2 \leqslant M_0'\|A^{\beta/2}w\|_2\|\nabla v\|_2. \tag{2.13b}$$

For $\sigma = \alpha/2 - 3/4$, $\beta \leqslant 3/2$, and $n \geqslant m$ there exist constants K_0 and K_0' such that

$$\|A^{-\gamma}Q_m(v\cdot\nabla)w\|_2 \leqslant \lambda_{m+1}^{-\sigma}K_0\|\nabla v\|_2\|A^{\beta/2}w\|_2 \tag{2.13c}$$

and

$$\|A^{-\gamma}Q_m(w\cdot\nabla)v\|_2 \leqslant \lambda_{m+1}^{-\sigma}K_0'\|A^{\beta/2}w\|_2\|\nabla v\|_2. \tag{2.13d}$$

In proving this we start with (2.1) which gives a constant $M_2 = M_1(\gamma/2, p, 2, \Omega)$ such that

$$\|A^{-\gamma}(v\cdot\nabla)w\|_2 \leqslant M_2\|v\cdot\nabla w\|_p \leqslant M_2\|v\|_{rp}\|\nabla w\|_{sp} \tag{2.14a}$$

where $p = 6/(4\gamma+3)$. We want $rp = 6$ so that by (2.1) $\|v\|_{rp} \leqslant M_3\|\nabla v\|_2$ for $M_3 = M_1(1/2, 2, rp, \Omega)$, which means that $r = 4\gamma+3$. Then $s = (4\gamma+3)/(4\gamma+2)$ so that $sp = 3/(2\gamma+1)$. From (2.1) there exists an M_4 such that $\|\nabla w\|_{sp} \leqslant M_4\|A^{(\beta-1)/2}(\nabla w)\|_2 = M_4\|A^{(\beta-1)/2}A^{1/2}w\|_2 = M_4\|A^{\beta/2}w\|_2$ provided that $sp = 3/(2\gamma+1) \leqslant 6/(3-2(\beta-1))$ which holds provided that $4\gamma+2 \geqslant 3-2\beta+2$ or $2\alpha-2\beta+2 \geqslant 3-2\beta+2$ which requires that $\alpha \geqslant 3/2$. Thus we obtain (2.13a) with $M_0 = M_2M_3M_4$ for $\beta \geqslant 1$.

For (2.13c) we modify (2.14a) by looking for ω such that

$$\|A^{-\omega}(v\cdot\nabla)w\|_2 \leqslant M_2'\|v\cdot\nabla w\|_p \leqslant M_2'\|v\|_{rp}\|\nabla w\|_{sp} \tag{2.14b}$$

where now $p = 6/(4\omega+3)$. We again want $rp = 6$ so now $r = 4\omega+3$, so that $sp = 3/(2\omega+1)$; by (2.1) we have that $\|\nabla w\|_{sp} \leqslant M_5\|A^{(\beta-1)/2}(\nabla w)\|_2 = M_5\|A^{(\beta-1)/2}A^{1/2}w\|_2 = M_5\|A^{\beta/2}w\|_2$ provided that $\beta \geqslant 1$ and that $sp = 3/(2\omega+1) \leqslant 6/(3-2(\beta-1))$ which leads to the condition $0 \leqslant 3/4 - \beta/2 \leqslant \omega$ (and we can take $\omega = 0$ if $\beta \geqslant 3/2$). With this we obtain (2.13c) by Poincaré with $K_0 = M_2' M_3 M_5$ since $\gamma - \omega = \sigma$.

For (2.13b) we have that

$$\left\|A^{-\gamma}(w\cdot\nabla)v\right\|_2 \leqslant M_6\|w\cdot\nabla v\|_p \leqslant M_6\|w\|_{rp}\|\nabla v\|_{sp} \tag{2.14c}$$

where $p = \max\{1, 6/(4\gamma+3)\}$ and if $p = 1$ then we take $r = s = 2$ so that (2.13b) and (2.13d) hold by Poincaré. Otherwise we now want $sp = 2$ which says that $s = (4\gamma+3)/3$ so that $r = (4\gamma+3)/(4\gamma)$ and thus $rp = 3/(2\gamma)$. We want for some M_7 that $\|w\|_{rp} \leqslant M_7\|A^{\beta/2}w\|_2$ or $3/(2\gamma) \leqslant 6/(3-2\beta)$ or $3 - 2\beta \leqslant 4\gamma$ which again leads to the condition $\alpha \geqslant 3/2$. Thus we obtain (2.13b) with $M_0' = M_6 M_7$.

For (2.13d) with $1 \leqslant \beta \leqslant \alpha$ we modify (2.14c) by looking for ω such that

$$\left\|A^{-\omega}(w\cdot\nabla)v\right\|_2 \leqslant M_8\|w\cdot\nabla v\|_p \leqslant M_8\|w\|_{rp}\|\nabla v\|_{sp} \tag{2.14d}$$

where as in (2.14b) $p = 6/(4\omega+3)$. We now want $sp = 2$ which says that $s = (4\omega+3)/3$ so that $r = (4\omega+3)/(4\omega)$ and thus $rp = 3/(2\omega)$. We want for some M_9 that $\|w\|_{rp} \leqslant M_9\|A^{\beta/2}w\|_2$ or $3/(2\omega) \leqslant 6/(3-2\beta)$ or $3 - 2\beta \leqslant 4\omega$; this again for $1 \leqslant \beta \leqslant 3/2$ leads to the condition $0 \leqslant 3/4 - \beta/2 \leqslant \omega$ (for which again we can take $\omega = 0$ if $\beta \geqslant 3/2$), and thus we have (2.13d) by Poincaré with $K_0' = M_8 M_9$. This finishes the proof of Lemma 6.

Next we consider the case $\beta \geqslant \alpha$, for which we will prove the following:

Lemma 7. *Let $\beta \geqslant \alpha \geqslant 3/2$ and let $\gamma = (\beta-\alpha)/2$. Then for $v \in H^{2\gamma+1}(\Omega)$ and $w \in H^{\beta/2}(\Omega)$ we have for a constant M_0'' that*

$$\left\|A^{\gamma}(w\cdot\nabla)v\right\|_2 \leqslant M_0''\left\|A^{\gamma+1/2}v\right\|_2\left\|A^{\beta/2}w\right\|_2^2 \tag{2.15a}$$

and that

$$\left\|A^{\gamma}(v\cdot\nabla)w\right\|_2 \leqslant M_0''\left\|A^{\gamma+1/2}v\right\|_2\left\|A^{\beta/2}w\right\|_2^2. \tag{2.15b}$$

To prove this, we have for the appropriate tensor product $\otimes$ that $v\cdot\nabla w = \operatorname{div}(v\otimes w)$ so that for integer values of $\gamma+1/2$ we have that

$$\begin{aligned}\left\|A^{\gamma}(v\cdot\nabla)w\right\|_2 &= \left\|A^{\gamma+1/2}\big(A^{-1/2}\operatorname{div}\big)(v\otimes w)\right\|_2\\ &= \left\|\big(A^{-1/2}\operatorname{div}\big)A^{\gamma+1/2}(v\otimes w)\right\|_2\\ &\leqslant \left\|A^{\gamma+1/2}(v\otimes w)\right\|_2.\end{aligned} \tag{2.16}$$

The leading terms of $A^{\gamma+1/2}(v\otimes w)$ are by Leibniz $(A^{\gamma+1/2}v)\otimes w$ and $v\otimes(A^{\gamma+1/2}w)$. We have that $\|v\otimes A^{\gamma+1/2}w\|_2 \leqslant \|v\|_{2r}\|A^{\gamma+1/2}w\|_{2s}$. Note that $\beta/2 - \gamma + 1/2 = (\alpha-1)/2$ so that $\|A^{\gamma+1/2}w\|_{2s} \leqslant M_{10}\|A^{(\alpha-1)/2}(A^{\gamma+1/2}w)\|_2 = M_{10}\|A^{\beta/2}w\|_2^2$ for $2s = 6/[3-2(\alpha-1)]$ or $s = 3/[3-2(\alpha-1)]$, therefore $r = s/(s-1) = 3/[2(\alpha-1)]$ so that $2r = 3/(\alpha-1)$. We have that $\|v\|_{3/(\alpha-1)} \leqslant M_{11}\|\nabla v\|_2$ if $3/(\alpha-1) \leqslant 6/(3-2) = 6$ which holds provided that $1/(\alpha-1) \leqslant 2$ or $\alpha \geqslant 3/2$. Thus for some M_{12} we have that $\|v\|_{2r} \leqslant M_{12}\|A^{\gamma+1/2}v\|_2$ by Poincaré. We thus have that

$$\left\|v\otimes A^{\gamma+1/2}w\right\|_2 \leqslant M_{10}M_{12}\|\nabla v\|_2\left\|A^{\beta/2}w\right\|_2.$$

We also have that $\|(A^{\gamma+1/2}v)\otimes w\|_2 \leqslant \|(A^{\gamma+1/2}v)\|_{2a}\|w\|_{2b}$; we need $\|w\|_{2b} \leqslant M_{13}\|A^{\beta/2}w\|_2$, but since $\beta \geqslant \alpha \geqslant 3/2$ we have that $\beta/2 \geqslant \alpha \geqslant 3/4$ so we can take $b = 2b = \infty$. Then $a = 1$ so that

$2a = 2$ and so $\|(A^{\gamma+1/2}v) \otimes w\|_2 \leqslant M_{14}\|A^{\gamma+1/2}v\|_2\|A^{\beta/2}w\|_2^2$. For the j middle terms in the Leibniz expansion we use Hölder to get the same bounds as above for constants M_{15}^j. By interpolation for noninteger $\gamma + 1/2$ we get (2.15a), and by switching the roles of v and w we get (2.15b). This completes the proof of Lemma 7.

Next we prove Theorem 3. From (2.7) we obtain that

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu\lambda_1}\int_0^t \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)}\,ds. \tag{2.17}$$

Let $f_t(s) = \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)}$ then note that $|f_t| \leqslant \|g(s)\|_2^2 \in L^1(0,\infty)$ and that $f_t(s) \to 0$ as $t \to \infty$ for each s; given $\epsilon > 0$ choose N large enough such that $\int_N^\infty \|g(s)\|_2^2\,ds < \epsilon/2$ then

$$\begin{aligned} 0 &\leqslant \int_0^t \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)}\,ds \\ &\leqslant \int_0^N \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)}\,ds + \int_N^\infty \|g(s)\|_2^2 s < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned} \tag{2.18}$$

for large enough t by the Dominated Convergence Theorem. From (2.18) and (2.17) we see that $\|u(t)\|_2^2 \to 0$ as $t \to \infty$. By a standard interpolation inequality we have that

$$\|u(t)\|_{\beta,2} \leqslant \|u(t)\|_{\theta\beta,2}^{1/\theta}\|u(t)\|_2^{1-1/\theta} \tag{2.19}$$

where $\|u\|_{\gamma,2}$ denotes the norm in the Sobolev space $W^{\gamma,2}(\Omega)$. Since $\|A^{\beta/2}u\|_2$ is a norm equivalent to $\|u\|_{\beta,2}$, and since each of the norms $\|u(t)\|_{\theta\gamma,2}$ was shown to be uniformly bounded for all $\theta\gamma \geqslant 0$ in [1, Section 2], we thus have Theorem 3 from (2.17), (2.18), and (2.19).

The following result, of some interest in its own right, will be key to proving Theorem 4. We need it rather than depending on the lemmas that appeared in [15,22] (see also [14, Lemma 1.1, Chapter III]), in that they in contrast will lead to exponential dependence on the data; note in particular the use of $\Gamma' \equiv e^{(\gamma_m+\Gamma)T_1}$ for a constant Γ depending on the size of the data in [14, Lemma 1.1, Chapter III].

Lemma 8. *For any $\lambda > 0$ we have that the solution u of* (1.1) *satisfies*

$$\begin{aligned} &\nu\int_0^t e^{-\lambda(t-s)}\|\nabla u\|_2^2\,ds + 2\mu\int_0^t e^{-\lambda(t-s)}\|Q_m A^{\alpha/2}u\|_2^2\,ds \\ &\quad\leqslant \|u_0\|_2^2 e^{-\lambda t} + U_g^2 + \frac{L_g^2}{\nu\lambda_1\lambda}. \end{aligned} \tag{2.20}$$

To prove this, we multiply both sides of (2.6) by $e^{\lambda s}$ and add $\lambda\|u\|_2^2 e^{\lambda s}$ to both sides to obtain

$$\begin{aligned} &\frac{d}{ds}\left(\|u\|_2^2 e^{\lambda s}\right) + \nu\|\nabla v\|_2^2 e^{\lambda s} + 2\mu\|Q_m A^{\alpha/2}u\|_2^2 e^{\lambda s} \\ &\quad\leqslant \lambda\|u\|_2^2 e^{\lambda s} + \frac{1}{\nu\lambda_1}\|g\|_2^2 e^{\lambda s}. \end{aligned} \tag{2.21}$$

Now integrate both sides of (2.21) from 0 to t to obtain

$$\|u\|_2^2 e^{\lambda t} + \nu \int_0^t e^{\lambda s} \|\nabla u\|_2^2 + 2\mu \int_0^t e^{\lambda s} \left\| Q_m A^{\alpha/2} u \right\|_2^2 ds$$
$$\leqslant \|u_0\|_2^2 + \lambda \int_0^t \|u\|_2^2 e^{\lambda s}\, ds + \frac{1}{\nu\lambda_1} \int_0^t e^{\lambda s} \|g\|_2^2\, ds. \tag{2.22}$$

Multiplying both sides of (2.22) by $e^{-\lambda t}$ we obtain

$$\|u\|_2^2 + \nu \int_0^t e^{-\lambda(t-s)} \|\nabla u\|_2^2\, ds + 2\mu \int_0^t e^{-\lambda(t-s)} \left\| Q_m A^{\alpha/2} u \right\|_2^2 ds$$
$$\leqslant \|u_0\|_2^2 e^{-\lambda t} + \lambda \int_0^t \|u\|_2^2 e^{-\lambda(t-s)}\, ds + \frac{1}{\nu\lambda_1} \int_0^t e^{-\lambda(t-s)} \|g\|_2^2\, ds. \tag{2.23}$$

Using (2.8) and (2.12), together with

$$\int_0^t e^{-\lambda(t-s)}\, ds = \int_0^t e^{-\lambda s}\, ds \leqslant \int_0^\infty e^{-\lambda s}\, ds \leqslant \frac{1}{\lambda} \tag{2.24}$$

we obtain Lemma 8.

With this result we close our discussion of preliminary observations and results.

3. Proof of Theorem 1

Let $w_N = u - u_N$ then subtracting (1.2a) from (1.1a) we obtain the following equations for w_N:

$$(w_N)_t + \nu A w_N + \mu A_\varphi w_N + P_N(u_N \cdot \nabla) w_N + P_N(w_N \cdot \nabla) u$$
$$= G_N + \nabla P_N + Q_N(u \cdot \nabla) u \tag{3.1}$$

where $G_N = g - g_N$ and $P_N = P - P_N$.

Taking the inner product of both sides of (3.1) with $A^\beta w_N$ we obtain

$$\frac{1}{2}\frac{d}{dt} \left\| A^{\beta/2} w_N \right\|_2^2 + \left(\nu A w_N + \mu A_\varphi w_N, A^\beta w_N\right) = \left(V, A^\beta w_N\right) \tag{3.2}$$

where $V =$ the right-hand side of (3.1) minus the nonlinear terms of the left-hand side. We assume for simplicity that $\mu \leqslant \nu$ (otherwise replace μ by $\mu_0 \equiv \min\{\mu, \nu\}$) then since the operators A and A_φ are positive we have that

$$\left(\nu A + \mu A_\varphi w_N, A^\beta w_N\right) \geqslant \mu\left((A + A_\varphi) w_N, A^\beta w_N\right)$$
$$\geqslant \mu\left(\left(P_m A + Q_m A^\alpha\right) w_N, A^\beta w_N\right). \tag{3.3}$$

Now $P_m A + Q_m A^\alpha = N^{\alpha-1} A^\alpha$ where $N = P_m A^{-1} + Q_m I$.

The smallest eigenvalue of $N^{\alpha-1}$ is $1/\lambda_m^{\alpha-1}$, thus from (3.3)

$$\begin{aligned}\big(\nu A+\mu A_\varphi w_N, A^\beta w_N\big) &\geqslant \mu\big(N^{\alpha-1}A^\alpha w_N, A^\beta w_N\big)\\ &= \mu\big(N^{\alpha-1}A^{(\alpha+\beta)/2}w_N, A^{(\alpha+\beta)/2}w_N\big)\\ &\geqslant \frac{\mu}{\lambda_m^{\alpha-1}}\big\|A^{(\alpha+\beta)/2}w_N\big\|_2^2. \end{aligned} \tag{3.4}$$

Meanwhile $(\nabla P_N, A^\beta w_N)=0$ since $\nabla\cdot A^\beta w_N = A^\beta(\nabla\cdot w_N)=0$ on a periodic box, while for $V_1 = V-\nabla P_N$ we have $(V_1, A^\beta w_N) = (A^{-\gamma}V_1, A^{(\alpha+\beta)/2}w_N)$ for $\gamma=(\alpha-\beta)/2$. By Young's inequality

$$\big(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}w_N\big) \leqslant \frac{2\lambda_m^{\alpha-1}}{\mu}\sum_{i=1}^4\big\|A^{-\gamma}v_i\big\|_2^2 + \frac{\mu}{2\lambda_m^{\alpha-1}}\big\|A^{(\alpha+\beta)/2}w_N\big\|_2^2 \tag{3.5}$$

where the v_i, $i=1,\dots,4$, are the terms of V_1. Combining (3.2), (3.4), and (3.5), collecting the terms of $\|A^{(\alpha+\beta)/2}w_N\|_2^2$, and multiplying by 2 we have

$$\begin{aligned}&\frac{d}{dt}\big\|A^{\beta/2}w_N\big\|_2^2 + \frac{\mu}{\lambda_m^{\alpha-1}}\big\|A^{(\alpha+\beta)/2}w_N\big\|_2^2\\ &\quad\leqslant \frac{4\lambda_m^{\alpha-1}}{\mu}\big(\big\|A^{-\gamma}P_N(u_N\cdot\nabla)w_N\big\|_2^2 + \big\|A^{-\gamma}P_N(w_N\cdot\nabla)u\big\|_2^2\\ &\qquad + \big\|A^{-\gamma}G_N\big\|_2^2 + \big\|A^{-\gamma}Q_N(u\cdot\nabla)u\big\|_2^2\big). \end{aligned}\tag{3.6}$$

By Lemmas 6 and 7, with $v=u_N$ or u and $w=w_N$, we have for $C_0=\max\{M_0, M_0'\}$ that

$$\begin{aligned}\frac{d}{dt}\big\|A^{\beta/2}w_N\big\|_2^2 &\leqslant \frac{4\lambda_m^{\alpha-1}C_0}{\mu}\big[\big\|A^\theta u_N\big\|_2^2 + \big\|A^\theta u\big\|_2^2\big]\big\|A^{\beta/2}w_N\big\|_2^2\\ &\quad + \frac{4\lambda_m^{\alpha-1}}{\mu}\left[\frac{1}{\lambda_1^\gamma}\|G_N\|_2^2 + \big\|A^{-\gamma}Q_N(u\cdot\nabla)u\big\|_2^2\right]\end{aligned}\tag{3.7}$$

where we have discarded the term $\mu\lambda_m^{1-\alpha}\|A^{(\alpha+\beta)/2}w_N\|_2^2$, used Poincaré on $\|A^{-\gamma}G_N\|_2^2$, and used the fact that P_N is an orthogonal projection; here $\theta=1/2$ if $\alpha\geqslant\beta$ and $\theta=\gamma+1/2$ if $\alpha<\beta$.

Integrating on $(0,t)$ and using Gronwall's inequality we have for $W_N(t)$ as in (1.3c) that

$$\big\|A^{\beta/2}w_N(t)\big\|_2^2 \leqslant W_N(t)\exp\left(\frac{4\lambda_m^{\alpha-1}C_0}{\mu}\int_0^t\big[\big\|A^\theta u_N\big\|_2^2 + \big\|A^\theta u\big\|_2^2\big]\,ds\right). \tag{3.8}$$

For $\beta>\alpha$ we obtain from (3.8) and the regularity results in [1, Section 2] (which show that $\int_0^t\|A^\theta u\|_2\,ds \leqslant C_1(U_g,T,\theta)$ where C_1 is a polynomial of degree depending on θ) that

$$\big\|A^{\beta/2}w_N(t)\big\|_2^2 \leqslant W_N(t)\exp\left(\frac{4\lambda_m^{\alpha-1}C_0C_1T}{\mu}\right) \tag{3.9}$$

where again $W_N(t)$ is as in (1.3c).

Let $C_2(u_0,\beta)$ be a bound from [1, Section 2] on $\|A^{\beta/2}u\|_2^2$, then Theorem 1 for $\beta>\alpha$ follows from (3.9), the integrability from the energy inequality of $\|A^{-\gamma}(u\cdot\nabla)u\|_2^2 \leqslant M_0\|A^{\beta/2}u\|_2^2\|\nabla u\|_2^2 \leqslant M_0C_2\|\nabla u\|_2^2$, the integrability of $\|G_N\|_2^2 \leqslant 2\|g\|_2^2$, and the Dominated Convergence Theorem.

For $\beta\leqslant\alpha$ so that $\gamma=1/2$, we use (2.6) to obtain in standard fashion

$$\int_0^t \|A^{1/2}u\|_2^2\,ds \leqslant \frac{1}{\nu^2\lambda_1}\int_0^t \|g\|_2^2\,ds \tag{3.10}$$

and

$$\int_0^t \|A^{1/2}u_N\|_2^2\,ds \leqslant \frac{1}{\nu^2\lambda_1}\int_0^t \|g_N\|_2^2\,ds \leqslant \frac{1}{\nu^2\lambda_1}\int_0^t \|g\|_2^2\,ds \tag{3.11}$$

so that from (3.8)

$$\|A^{\beta/2}w_N(t)\|_2^2 \leqslant W_N(t)\exp\left(8\mu^{-1}\lambda_m^{\alpha-1}C_0(\lambda_1\nu)^{-2}\int_0^t \|g\|_2^2\,ds\right). \tag{3.12}$$

Using the discussion involving the Dominated Convergence Theorem and C_2 from the above we see that the convergence as $N\to\infty$ in (3.12) is uniform on $[0,T]$, proving Theorem 1 in the case $\beta\leqslant\alpha$, and thus we conclude this section.

4. Proofs of Theorems 4 and 5

We apply Q_n to both sides of (3.1) and take the inner product with $A^{\beta}Q_nw_N$, noting that $Q_nA_\varphi v = Q_nA^\alpha v$ since $n\geqslant m$ and that $(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}Q_nw_N) = (A^{-\gamma}V_1, A^{(\alpha+\beta)/2}Q_n^2w_N) = (A^{-\gamma}Q_nV_1, A^{(\alpha+\beta)/2}Q_nw_N)$. We then proceed as in (3.6), (3.7), only now we use (2.13c), (2.13d) of Lemma 6, and Young's inequality in a similar way to obtain for $1\leqslant\beta\leqslant 3/2\leqslant\alpha$ and for $C_1=\max\{K_0,K_0'\}$ that

$$\begin{aligned}
&\frac{d}{dt}\|A^{\beta/2}Q_nw_N\|_2^2 + \mu\|A^{(\alpha+\beta)/2}Q_nw_N\|_2^2 \\
&\quad\leqslant \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}\big[\|A^{1/2}u_N\|_2^2 + \|A^{1/2}u\|_2^2\big]\|A^{\beta/2}w_N\|_2^2 \\
&\qquad + \frac{4}{\mu}\|A^{-\gamma}Q_N(u\cdot\nabla)u\|_2^2 + \frac{4}{\mu\lambda_1^\gamma}\|Q_nG_N\|_2^2
\end{aligned} \tag{4.1}$$

where since $\alpha\geqslant\beta$ we can take $\theta=1/2$ and where we note that we can assume that $N>n$; note also we have not included the term $\lambda_m^{\alpha-1}$ when using Young's inequality since we will not use the operator $N^{\alpha-1}$ in this section. Set

$$U_N = \|A^{1/2}u_N\|_2^2 + \|A^{1/2}u\|_2^2 \tag{4.2}$$

and

$$F_{Q,N}(t) = \frac{4}{\mu}\|Q_nG_N\|_2^2 + \frac{4}{\lambda_1^\gamma}\|A^{-\gamma}Q_N(u\cdot\nabla)u\|_2^2 \tag{4.3}$$

then applying Poincaré to (4.1) we have that

$$\begin{aligned}
&\frac{d}{dt}\|A^{\beta/2}Q_n w_N\|_2^2 + \mu\lambda_{n+1}^{\alpha}\|A^{\beta/2}Q_n w_N\|_2^2 \\
&\quad\leqslant \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}U_N\|A^{\beta/2}w_N\|_2^2 + F_{Q,N}(t) \\
&\quad\leqslant \frac{4C_1}{\mu}U_N\big(\lambda_{n+1}^{-2\sigma}\|A^{\beta/2}P_n w_N\|_2^2 + \lambda_{n+1}^{-2\sigma}\|A^{\beta/2}Q_n w_N\|_2^2\big) + F_{Q,N}(t)
\end{aligned} \tag{4.4}$$

which in a DM approach we could write as

$$\begin{aligned}
&\frac{d}{dt}\|A^{\beta/2}Q_n w_N\|_2^2 + \left(\mu\lambda_{n+1}^{\alpha} - \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}U_N\right)\|A^{\beta/2}Q_n w_N\|_2^2 \\
&\quad\leqslant \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}U_N\|A^{\beta/2}P_n w_N\|_2^2 + F_{Q,N}(t)
\end{aligned} \tag{4.5}$$

or

$$\frac{d}{dt}\|A^{\beta/2}Q_n w_N\|_2^2 + a(t)\|A^{\beta/2}Q_n w_N(t)\|_2^2 \leqslant b(t) \tag{4.6}$$

where

$$a = \mu\lambda_{n+1}^{\alpha} - \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}U_N \tag{4.7a}$$

and

$$b = \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}U_N\|A^{\beta/2}P_n w_N\|_2^2 + F_{Q,N}(t). \tag{4.7b}$$

Alternatively, by Poincaré we have

$$\lambda_{n+1}^{\alpha-1}\|Q_n A^{1/2}u\|_2^2 \leqslant \|Q_n A^{(\alpha-1)/2}A^{1/2}u\|_2^2 = \|Q_n A^{\alpha/2}u\|_2^2 \tag{4.8}$$

so that, from Lemma 8 with $\lambda = \mu\lambda_{n+1}^{\alpha}$,

$$\begin{aligned}
\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}}\|Q_n A^{1/2}u\|_2^2\,ds &\leqslant \frac{1}{2\mu\lambda_{n+1}^{\alpha-1}}\left[\|u_0\|_2^2 + U_g^2 + \frac{L_g^2}{\mu\nu\lambda_1\lambda_{n+1}^{\alpha}}\right] \\
&\leqslant \frac{1}{2\mu\lambda_{n+1}^{\alpha-1}}\left[2\|u_0\|_2^2 + \left(1 + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\right] \\
&\leqslant \frac{1}{\mu\lambda_{n+1}^{\alpha-1}}\left(3 + 2\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2.
\end{aligned} \tag{4.9}$$

Meanwhile $\|P_n A^{1/2}u\|_2^2 \leqslant \lambda_n\|u\|_2^2 \leqslant \lambda_n U_g$ so that

$$\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} \left\| P_n A^{1/2} u \right\|_2^2 ds \leqslant \lambda_n U_g \int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} dt$$
$$\leqslant \lambda_n U_g \mu^{-1} \lambda_{n+1}^{-\alpha} \leqslant (\mu\lambda_{n+1})^{-(\alpha-1)}(2+\eta)\left(\frac{L_g}{\nu\lambda_1}\right)^2. \quad (4.10)$$

Combining (4.9) and (4.10) we have that

$$\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} \|\nabla u\|_2^2 ds \leqslant \frac{1}{\mu\lambda_{n+1}^{\alpha-1}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2. \quad (4.11)$$

We note that (4.11) holds with u replaced by u_N, since Lemma 8 holds with u_N replacing u, and since $\|u_N(0)\|_2 \leqslant \|u_0\|_2$ and $\|g_N\|_2 \leqslant \|g\|_2$. Hence

$$\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} U_N\, ds \leqslant \frac{2}{\mu\lambda_{n+1}^{\alpha-1}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2. \quad (4.12)$$

Integrating the differential inequality (4.4) we obtain

$$\begin{aligned}\left\| A^{\beta/2} Q_n w_N \right\|_2^2 \leqslant{}& \left\| A^{\beta/2} Q_n w_N(0) \right\|_2^2 e^{-\mu\lambda_{n+1}^{\alpha} t} \\ &+ \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu} \int_0^t e^{-\mu\lambda_{n+1}^{\alpha}(t-s)} U_N \left\| A^{\beta/2} Q_n w_N \right\|_2^2 ds \\ &+ \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu} \int_0^t e^{-\mu\lambda_{n+1}^{\alpha}(t-s)} U_N \left\| A^{\beta/2} P_n w_N \right\|_2^2 ds \\ &+ \int_0^t e^{-\mu\lambda_{n+1}^{\alpha}(t-s)} F_{Q,N}(s)\, ds. \end{aligned} \quad (4.13)$$

Setting $\rho_P(t) = \sup_{0\leqslant s\leqslant t} \|A^{\beta/2} P_n w_N\|_2^2$ and setting $\rho_Q(t) = \sup_{0\leqslant s\leqslant t} \|A^{\beta/2} Q_n w_N(s)\|_2^2$, we have from (4.12) and (4.13) that

$$\begin{aligned}\left\| A^{\beta/2} Q_n w_N \right\|_2^2 \leqslant{}& \left\| A^{\beta/2} Q_n w_N(0) \right\|_2^2 \\ &+ \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}\left(\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} U_N\, ds\right)\rho_Q(t) \\ &+ \frac{4C_1\lambda_{n+1}^{-2\sigma}}{\mu}\left(\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} U_N\, ds\right)\rho_P(t) \\ &+ \int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} F_{Q,N}(s)\, ds \end{aligned}$$

$$\leqslant \left\| A^{\beta/2} Q_n w_N(0) \right\|_2^2 + \int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} F_{Q,N}(s)\, ds$$
$$+ \frac{8C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_Q(t)$$
$$+ \frac{8C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P(t). \tag{4.14}$$

Set $\mathcal{F}_{Q,N}(t) = \sup_{0\leqslant s\leqslant t} \int_0^s e^{-\mu(s-\tau)\lambda_{n+1}^{\alpha}} F_{Q,N}(\tau)\, d\tau$, then (4.14) holds with $\mathcal{F}_{Q,N}$ replacing $\int_0^t e^{-\mu(t-s)\lambda_{n+1}^{\alpha}} F_{Q,N}(s)\, ds$ on the right-hand side so that

$$\rho_Q(t) \leqslant \left\| A^{\beta/2} Q_n w_N \right\|_2^2 + \mathcal{F}_{Q,N}(t)$$
$$+ \frac{8C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_Q(t)$$
$$+ \frac{8C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P(t); \tag{4.15}$$

hence if

$$\frac{8C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \leqslant \frac{1}{2} \tag{4.16}$$

which requires that

$$\lambda_{n+1}^{2\alpha-5/2} \geqslant \frac{16C_1}{\mu^2} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{4.17}$$

then

$$\rho_Q(t) \leqslant 2\left\| A^{\beta/2} Q_n w_N(0) \right\|_2^2 + 2\mathcal{F}_{Q,N}(t)$$
$$+ \frac{16C_1}{\mu^2 \lambda_{n+1}^{2\alpha-5/2}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P(t). \tag{4.18}$$

Thus by choosing n large enough we obtain linear dependence of ρ_Q on all convergence factors, and there are no exponential terms except as may appear in estimates of $\rho_P(t)$. This finishes the proof of Theorem 4, using an integral-equation variant of (4.6).

Now we use the above results to obtain a bound on $\|A^{\beta/2} P_n w_N\|_2^2$, and thus establish Theorem 5. First we apply P_n to both sides of (3.1) and take the inner product with $A^{\beta/2} P_n w_N$ to obtain in similar fashion to the calculations (3.2)–(3.6) that

$$\frac{1}{2}\frac{d}{dt}\left\| A^{\beta/2} P_n w_N \right\|_2^2 + \nu \left\| A^{\frac{1+\beta}{2}} P_n w_N \right\|_2^2$$
$$\leqslant C_0\big(\|\nabla u_N\|_2 + \|\nabla u\|_2\big)\left\| A^{\frac{\alpha+\beta}{2}} P_n w_N \right\|_2 + \frac{1}{\lambda_1^{1/2}} \|G_N\|_2 \left\| A^{\frac{1+\beta}{2}} w_N \right\|_2 \tag{4.19}$$

where we note that $P_n Q_N = 0$ and that (4.19) holds for any $\alpha \geqslant 3/2$.

Set $\alpha = 3/2$ then $\|A^{\frac{\alpha+\beta}{2}} P_n w_N\|_2 = \|A^{\frac{(3/2)+\beta}{2}} P_n w_N\|_2 \leqslant \lambda_n^{1/4} \|A^{\frac{1+\beta}{2}} P_n w_N\|_2$; applying Young's inequality we have

$$\frac{d}{dt}\left\|A^{\beta/2}P_n w_N\right\|_2^2 + \nu\left\|A^{\frac{1+\beta}{2}}P_n w_N\right\|_2^2 \leqslant \frac{3}{\nu\lambda_1}\|G_N\|_2^2 + \frac{3\lambda_n^{1/2}C_0}{\nu}U_N\left\|A^{\beta/2}Q_n w_N\right\|_2^2 + \frac{3\lambda_n^{1/2}C_0}{\nu}U_N\left\|A^{\beta/2}P_n w_N\right\|_2^2. \tag{4.20}$$

Noting that $\|A^{\frac{1+\beta}{2}} P_n w_N\|_2^2 \leqslant \lambda_1 \|A^{\beta/2} P_n w_N\|_2^2$ we integrate the differential inequality (4.20) to obtain

$$\begin{aligned}\left\|A^{\beta/2}P_n w_N\right\|_2^2 \leqslant{}& \left\|A^{\beta/2}P_n w_N\right\|_2^2 e^{-\nu\lambda_1 t} + \frac{3}{\nu\lambda_1}\int_0^t e^{-\nu\lambda_1(t-s)}\|G_N\|_2^2\,ds \\ &+ \frac{3\lambda_n^{1/2}C_0}{\nu}\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\rho_Q(s)\,ds \\ &+ \frac{3\lambda_n^{1/2}C_0}{\nu}\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\rho_P(s)\,ds.\end{aligned} \tag{4.21}$$

Combining (4.17) and (4.18), and noting that $\mathcal{F}_{Q,N}(s) \leqslant \mathcal{F}_{Q,N}(t)$ for $s \leqslant t$ we have that

$$\rho_Q(s) \leqslant 2\left\|A^{\beta/2}Q_n w_N(0)\right\|_2^2 + 2\mathcal{F}_{Q,N}(t) + \rho_P(s). \tag{4.22}$$

In similar fashion to (4.9)

$$\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\,ds \leqslant \frac{1}{\nu}(3+2\eta+\nu)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{4.23}$$

so combing (4.21)–(4.23) we have for

$$\mathcal{L}_{G_N} \equiv \sup_{0\leqslant s\leqslant t}\frac{3}{\nu\lambda_1}\int_0^s e^{-\nu\lambda_1(s-\tau)}\left\|G_N(\tau)\right\|_2^2\,d\tau \tag{4.24}$$

that

$$\begin{aligned}\left\|A^{\beta/2}P_n w_N\right\|_2^2 \leqslant{}& \left\|A^{\beta/2}P_n w_N(0)\right\|_2^2 + \frac{3}{(\nu\lambda_1)^2}\mathcal{L}_{G_N} \\ &+ \frac{6\lambda_n^{1/2}C_0(3+2\eta+\nu)}{\nu^2}\left(\frac{L_g}{\nu\lambda_1}\right)^2\left[\left\|A^{\beta/2}Q_n w_N(0)\right\|_2^2 + \mathcal{F}_{Q,N}(t)\right] \\ &+ \frac{6\lambda_n^{1/2}C_0}{\nu}\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\rho_P(s)\,ds.\end{aligned} \tag{4.25}$$

Neglecting the exponential term in the last term of (4.25) and taking the sup over $[0,t]$ on the left-hand side we obtain

$$\rho_P(t) \leqslant \left\| A^{\beta/2} P_n w_N \right\|_2^2 + \frac{3}{(\nu\lambda_1)^2} \mathcal{L}_{G_N} + \frac{6\lambda_n^{1/2} C_0 (3+2\eta+\nu)}{\nu^2} \left(\frac{L_g}{\nu\lambda_1} \right)^2 \left[\left\| A^{\beta/2} Q_n w_N(0) \right\|_2^2 + \mathcal{F}_{Q,N}(t) \right] + \frac{6\lambda_n^{1/2} C_0}{\nu} \int_0^t U_N(s) \rho_P(s)\, ds \tag{4.26}$$

from which by Gronwall we obtain, for $w_{N,0} \equiv w_N(0)$, for $G_0 = L_g/(\nu\lambda_1)$, and

$$W_{N,0}(t) \equiv \left\| A^{\beta/2} P_n w_{N,0} \right\|_2^2 + \frac{3\mathcal{L}_{G_N}}{(\nu\lambda_1)^2} + \frac{6\lambda_n^{1/2} C_0 (3+2\eta+\nu)}{\nu^2} G_0^2 \left(\left\| A^{\beta/2} Q_n w_{N,0} \right\|_2^2 + \mathcal{F}_{Q,N}(t) \right) \tag{4.27}$$

that

$$\rho_P(t) \leqslant W_{N,0}(t) \exp\left(\frac{6\lambda_n^{1/2} C_0}{\nu} \int_0^t U_N(s)\, ds \right) \leqslant W_{N,0}(t) \exp\left(\frac{12\lambda_n^{1/2} C_0}{\nu^2} \left[(1+\eta) G_0^2 + \frac{1}{\nu\lambda_1} \int_0^t \|g\|_2^2\, ds \right] \right) \tag{4.28}$$

and in the case of decaying turbulence we can replace $\int_0^t \|g\|_2^2\, ds$ by $\int_0^\infty \|g\|_2^2\, ds$. The estimate (4.28) grows with n, but only as a fractional power of it; in fact, since $\lambda_n \sim c\lambda_1 n^{2/3}$, the dependence on n is like $n^{1/3}$. This completes the proof of Theorem 5.

5. Continuous dependence on data

Suppose we have two solutions $v(t)$ and $u(t)$ to (1.1) with forcing data $f(t)$ and $g(t)$ and pressure terms p_1 and p_2 respectively. Their difference $w(t) \equiv v(t) - u(t)$ satisfies an equation similar to (3.1), namely:

$$(w)_t + \nu A w + \mu A_\varphi w + (v \cdot \nabla) w + (w \cdot \nabla) u = G + \nabla P \tag{5.1}$$

where $G = f - g$ and $P = p_1 - p_2$. Using the same techniques that led to (4.6) and (4.7) with just a slightly different use of Young's inequality (there is no term like $A^{-\gamma} Q_N (u \cdot \nabla) u$) we obtain for $n \geqslant m$ that $Q_n w$ satisfies

$$\frac{d}{dt} \left\| A^{\beta/2} Q_n w \right\|_2^2 + a(t) \left\| A^{\beta/2} Q_n w(t) \right\|_2^2 \leqslant b(t) \tag{5.2}$$

where for $U \equiv \|A^{1/2}u\|_2^2 + \|A^{1/2}v\|_2^2$

$$a = \mu\lambda_{n+1}^{\alpha} - \frac{3C_1\lambda_{n+1}^{-2\sigma}}{\mu}U \tag{5.3a}$$

and

$$b = \frac{3C_1\lambda_{n+1}^{-2\sigma}}{\mu}U\left\|A^{\beta/2}P_m w\right\|_2^2 + \frac{3}{\mu}\|Q_n G\|_2^2. \tag{5.3b}$$

Based on these estimates it is straightforward to adapt the techniques used to prove Theorem 4 to obtain the following result relating the continuous dependence of $Q_n w$ on its corresponding data and on $P_n w$:

Theorem 9. *Let L_g and η be as above, suppose that $\alpha \geqslant 3/2 \geqslant \beta \geqslant 1$, and assume that*

$$\lambda_{n+1}^{2\alpha-5/2} \geqslant \frac{12C_1}{\mu^2}\left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{5.4}$$

then if I is any interval over which $\|A^{\beta/2}P_n w(t)\|_2^2$ and $\|Q_n G(t)\|_2^2$ have suprema, then we have for a generic constant C_1 as above that

$$\sup_{t\in I}\left\|A^{\beta/2}Q_n w(t)\right\|_2^2 \leqslant 2\left\|A^{\beta/2}Q_n w(0)\right\|_2^2 + \frac{6}{\mu}\sup_{t\in I}\left\|Q_n G(t)\right\|_2^2 + \frac{12C_1}{\mu^2\lambda_{n+1}^{2\alpha-5/2}}\left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \sup_{t\in I}\left\|A^{\beta/2}P_n w(t)\right\|_2^2. \tag{5.5}$$

To in turn establish estimates on $\|A^{\beta/2}P_m w(t)\|_2^2$, we can straightforwardly adapt the techniques used to prove Theorem 5. We note again the connection to the Kolmogorov theory (deeper here because there is no $Q_N(u\cdot\nabla)u$ term) arising from the linear dependence on the data in (5.5).

6. Conclusion

The SEV and SVV models arose as attempts to overcome well-known weaknesses of standard LES techniques (see e.g. [4,5,23]), whose underlying assumptions, for example, are at odds with the findings in [4] that the local energy flux can be poorly correlated with the local energy dissipation rate. As early as the mid-seventies other weaknesses of standard eddy-viscosity models had been observed and in [25] Kraichnan argued alternatively for modeling the effects of the subgrid scales on the resolvable scales via a dissipative term whose coefficient must vary with the wavenumber k; if we let m above represent the boundary between subgrid-scale wavenumbers $k > m$ and resolvable wavenumbers $k < m$, then his spectral eddy-viscosity coefficient is minimal until an m_0 such that $m - m_0 \leqslant (1/8)m$ whereupon the SEV coefficient rises sharply to a cusp between m_0 and m. That the coefficients of SEV should follow an Arrhenius-like distribution as a function of the wavenumber was further explored in [7,8].

Following the theoretical results for SVV developed for conservation laws by Tadmor [31], and subsequent computational results [6], Karamanos and Karniadakis [23] applied SVV to 3D turbulence modeling as an approximation to SEV; also see Sirisup and Karniadakis [30] for an application of SVV to a POD model of 2D flow. The application of SVV in [23] is accomplished using second-order kernels for ready implementation in a standard finite-element framework, but the authors point out that hyperviscosity kernels can be used in their methodological framework if discontinuous Galerkin techniques are used. One of the goals in [5] is to empirically study the coefficient distribution of spectral hyperviscosity, and for a common value of the Kolmogorov constant an Arrhenius-like distribution is observed for what is essentially the ratio μ/ν. One can view the basic idea of SVV as using a smoothed-out Heaviside function as an approximation to the Arrhenius-like distribution. Our main goal here has been to study Galerkin convergence for a version of SVV that directly realizes the idea of ‘using hyperviscous dissipation that will affect only the high Fourier modes’.

Meanwhile, since (2.13c) and (2.13d) hold for all $1 \leqslant \beta \leqslant 3/2 \leqslant \alpha$, we can for the moment replace β by $3/2$ in the statement of Theorem 4 and use Poincaré for $\beta < 3/2$ to note that

$$\lambda_{n+1}^{3/2-\beta} \left\| A^{\beta/2} Q_n w(t) \right\|_2^2 \leqslant \left\| A^{3/4} Q_n w(t) \right\|_2^2. \tag{6.1}$$

Using (6.1) on the left-hand sides of the estimate (1.16) and dividing through by $\lambda_{m+1}^{\alpha-\beta}$ we obtain

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \left\| A^{\beta/2} Q_n w(t) \right\|_2^2 \\ &\quad\leqslant \frac{2}{\lambda_{n+1}^{3/2-\beta}} \left\| A^{3/4} Q_n w_N(0) \right\|_2^2 + \frac{2}{\lambda_{n+1}^{3/2-\beta}} \mathcal{F}_{Q,N}(t) \\ &\quad\quad + \frac{16C_1}{\mu^2 \lambda_{n+1}^{2\alpha+3/2-(\beta+5/2)}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu\lambda_{n+1}^{\alpha}}\right) \left(\frac{L_g}{\nu\lambda_1}\right)^2 \sup_{0\leqslant t\leqslant T} \left\| A^{3/4} P_n w(t) \right\|_2^2 \end{aligned} \tag{6.2}$$

where we replace β by $3/2$ in the definition of $\mathcal{F}_{Q,N}(t)$; note that (6.2) is a stronger result that reduces the size of our estimates.

It is natural to assume that similar techniques to those used in the proof of Theorem 4 are applicable to estimating bounds on the number of determining modes. There are a few differences in the arguments used, especially those which involve examining certain estimates asymptotically as $t \to \infty$, whereas here we necessarily consider convergence on entire intervals $[0, T]$ and $[0, \infty)$. As such we will address the topic of determining modes in a separate paper, along with results on the number of determining nodes ([18], [22], also see [14, Chapter III]); in the latter cases there are some significant differences in the arguments that need to be used.

We note that DM and IM results, the related results in Theorems 4 and 5, and estimates for the number of degrees of freedom are a measure of the complexity of the systems they study. Further results in this regard include lower bounds on the dimension of the attractor for the 2D NSE; see the discussion and references in [14] and [34]. Another interesting way to obtain a lower-bound estimate on the complexity is to provide upper bounds on the size of the nodal set for the vorticity, as was done in [26,27] for periodic solutions of the 2D NSE.

It would have been fairly straightforward to also include results on the convergence of the attractors and inertial manifolds of the Galerkin systems to those corresponding to (1.1), along the lines of the corresponding results in [17,34]. We hope to discuss these in a separate paper in which we will also seek to establish trajectory-tracking results in which trajectories of Galerkin solutions will be compared with trajectories on the attractor for (1.1). In any future exploration of spectrally-hyperviscous models, we expect that the sharpest results will be obtained by employing spectral-decomposition techniques similar to the ones developed in [1] as well as those appearing in these pages. In essence, this approach reflects an underlying philosophy of treating turbulence modeling as a multiscale phenomenon.

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